Everything You Ever Need to Know About Analysis
MAT 215

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September 3, 2024

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Chapter 1

The Real Numbers

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Theorem 1.1.1

There is no rational number whose square is 2.

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1.2.2 Functions

Definition 1.2.1: Definition 1.2.3.

Given two sets A and B, a function from A to B is a rule or mapping that takes each element $x \in A$ and associates with it a single element of B. In this case, we write $f : A \to B$. Given an element $x \in A$, the expression $f(x)$ is used to represent the element of B associated with x by f. The set A is called the domain of f. The range of f is not necessarily equal to B but refers to the subset of B given by $\{y \in B : y = f(x)$ for some $x \in A\}$.

1.2.3 Logic and Proofs

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Two real numbers a and b are equal if and only if for every real number $\epsilon > 0$ it follows that $|a - b| < \epsilon$.

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1.3 The Axiom of Completeness

1.3.1 An Initial Definition for $\mathbb{R}$

Axiom of Completeness. Every nonempty set of real numbers that is bounded above has a least upper bound.

1.3.2 Least Upper Bounds and Greatest Lower Bounds

Definition 1.3.1: Definition 1.3.1.

A set $A \subseteq \mathbb{R}$ is bounded above if there exists a number $b \in \mathbb{R}$ such that $a \leqslant b$ for all $a \in A$. The number b is called an upper bound for A.
Similarly, the set A is bounded below if there exists a lower bound $l \in \mathbb{R}$ satisfying $l \leqslant a$ for every $a \in A$.

Definition 1.3.2: Definition 1.3.2.

A real number s is the least upper bound for a set $A \subseteq \mathbb{R}$ if it meets the following two criteria:

(a) s is an upper bound for A;

(b) if b is any upper bound for A, then $s \leqslant b$.

Definition 1.3.3: Definition 1.3.4.

A real number a_0 is a maximum of the set A if a_0 is an element of A and $a_0 \geqslant a$ for all $a \in A$. Similarly, a number a_1 is a minimum of A if $a_1 \in A$ and $a_1 \leqslant a$ for every $a \in A$.

Lemma 1.3.1 Lemma 1.3.8.

Assume $s \in \mathbb{R}$ is an upper bound for a set $A \subseteq \mathbb{R}$. Then, $s = \sup A$ if and only if, for every choice of $\epsilon > 0$, there exists an element $a \in A$ satisfying $s - \epsilon < a$.

1.4 Consequences of Completeness

Theorem 1.4.1 Theorem 1.4.1 (Nested Interval Property).

For each $n \in \mathbb{N}$, assume we are given a closed interval $I_n = [a_n, b_n] = \{x \in \mathbb{R} : a_n \leqslant x \leqslant b_n\}$. Assume also that each I_n contains I_{n+1}. Then, the resulting nested sequence of closed intervals

$$I_1 \supseteq I_2 \supseteq I_3 \supseteq I_4 \supseteq \cdots$$

has a nonempty intersection; that is, $\bigcap_{n=1}^{\infty} I_n \neq \emptyset$.

1.4.1 The Density of $\mathbb{Q}$ in $\mathbb{R}$

Theorem 1.4.2 Theorem 1.4.2 (Archimedean Property).

(a) Given any number $x \in \mathbb{R}$, there exists an $n \in \mathbb{N}$ satisfying $n > x$.

(b) Given any real number $y > 0$, there exists an $n \in \mathbb{N}$ satisfying $1/n < y$.

Theorem 1.4.3 Theorem 1.4.3 (Density of $\mathbb{Q}$ in $\mathbb{R}$).

For every two real numbers a and b with $a < b$, there exists a rational number r satisfying $a < r < b$.

Corollary 1.4.1 Corollary 1.4.4.

Given any two real numbers $a < b$, there exists an irrational number t satisfying $a < t < b$.

1.4.2 The Existence of Square Roots

Theorem 1.4.4 Theorem 1.4.5.
There exists a real number $\alpha \in \mathbb{R}$ satisfying $\alpha^2 = 2$.

1.5 Cardinality

1.5.1 1-1 Correspondence

Definition 1.5.1: Definition 1.5.1.

A function $f : A \to B$ is **one-to-one** (1-1) if $a_1 \neq a_2$ in A implies that $f(a_1) \neq f(a_2)$ in B. The function f is **onto** if, given any $b \in B$, it is possible to find an element $a \in A$ for which $f(a) = b$.

Definition 1.5.2: Definition 1.5.2.

The set A has the same cardinality as B if there exists $f : A \to B$ that is $1 - 1$ and onto. In this case, we write $A \sim B$.

1.5.2 Countable Sets

Definition 1.5.3: Definition 1.5.5.

A set A is **countable** if $\mathbb{N} \sim A$. An infinite set that is not countable is called an **uncountable** set.

Theorem 1.5.1 Theorem 1.5.6.
(i) The set $\mathbb{Q}$ is countable. (ii) The set $\mathbb{R}$ is uncountable.

Theorem 1.5.2 Theorem 1.5.7.
If $A \subseteq B$ and B is countable, then A is either countable or finite.

Theorem 1.5.3 Theorem 1.5.8.

(a) If $A_1, A_2, \ldots A_m$ are each countable sets, then the union $A_1 \cup A_2 \cup \cdots \cup A_m$ is countable.

(b) If A_n is a countable set for each $n \in \mathbb{N}$, then $\bigcup_{n=1}^{\infty} A_n$ is countable.

1.6 Cantor's Theorem

Theorem 1.6.1 Theorem 1.6.1.
The open interval $(0, 1) = \{x \in \mathbb{R} : 0 < x < 1\}$ is uncountable.

1.6.1 Power Sets and Cantor's Theorem

Theorem 1.6.2 Theorem 1.6.2 (Cantor's Theorem).
Given any set A, there does not exist a function $f : A \to P(A)$ that is onto.

1.7 Epilogue

Chapter 2

Sequences and Series

2.1 Discussion: Rearrangements of Infinite Series

2.2 The Limit of a Sequence

Definition 2.2.1: Definition 2.2.1.

A sequence is a function whose domain is $\mathbb{N}$.

Definition 2.2.2: Definition 2.2.3 (Convergence of a Sequence).

A sequence (a_n) **converges** to a real number a if, for every positive number ϵ, there exists an $N \in \mathbb{N}$ such that whenever $n \geqslant N$ it follows that $|a_n - a| < \epsilon$.

Definition 2.2.3: Definition 2.2.4.

Given a real number $a \in \mathbb{R}$ and a positive number $\epsilon > 0$, the set

$$V_\epsilon(a) = \{x \in \mathbb{R} : |x - a| < \epsilon\}$$

is called the ϵ-neighborhood of a.

Definition 2.2.4: Definition 2.2.3B (Convergence of a Sequence: Topological Version).

A sequence (a_n) converges to a if, given any ϵ-neighborhood $V_\epsilon(a)$ of a, there exists a point in the sequence after which all of the terms are in $V_\epsilon(a)$. In other words, every ϵ-neighborhood contains all but a finite number of the terms of (a_n).

2.2.1 Quantifiers

Template for a proof that $(x_n) \to x$:

- "Let $\epsilon > 0$ be arbitrary."
- Demonstrate a choice for $N \in \mathbb{N}$. This step usually requires the most work, almost all of which is done prior to actually writing the formal proof.
- Now, show that N actually works.
- "Assume $n \geqslant N$."
- With N well chosen, it should be possible to derive the inequality $|x_n - x| < \epsilon$.

Theorem 2.2.1 Theorem 2.2.7 (Uniqueness of Limits).
The limit of a sequence, when it exists, must be unique.

2.2.2 Divergence

Definition 2.2.5: Definition 2.2.9.

A sequence that does not converge is said to **diverge**.

2.3 The Algebraic and Order Limit Theorems

Definition 2.3.1: Definition 2.3.1.

A sequence (x_n) is bounded if there exists a number $M > 0$ such that $|x_n| \leqslant M$ for all $n \in \mathbb{N}$.

Theorem 2.3.1 Theorem 2.3.2.
Every convergent sequence is bounded.

Theorem 2.3.2 Theorem 2.3.3 (Algebraic Limit Theorem).
Let $\lim a_n = a$, and $\lim b_n =$ b. Then,

(a) $\lim (ca_n) = ca$, for all $c \in \mathbb{R}$;

(b) $\lim (a_n + b_n) = a + b$;

(c) $\lim (a_n b_n) = ab$;

(d) $\lim (a_n / b_n) = a/b$, provided $b \neq 0$.

2.3.1 Limits and Order

Theorem 2.3.3 Theorem 2.3.4 (Order Limit Theorem).
Assume $\lim a_n = a$ and $\lim b_n = b$.

(a) If $a_n \geqslant 0$ for all $n \in \mathbb{N}$, then $a \geqslant 0$.

(b) If $a_n \leqslant b_n$ for all $n \in \mathbb{N}$, then $a \leqslant b$.

(c) If there exists $c \in \mathbb{R}$ for which $c \leqslant b_n$ for all $n \in \mathbb{N}$, then $c \leqslant b$. Similarly, if $a_n \leqslant c$ for all $n \in \mathbb{N}$, then $a \leqslant c$.

Theorem 2.3.4 Exercise 2.3.3 (Squeeze Theorem).
Show that if $x_n \leqslant y_n \leqslant z_n$ for all $n \in \mathbb{N}$, and if $\lim x_n = \lim z_n = l$, then $\lim y_n = l$ as well.

2.4 The Monotone Convergence Theorem and a First Look at Infinite Series

Definition 2.4.1: Definition 2.4.1.

A sequence (a_n) is **increasing** if $a_n \leqslant a_{n+1}$ for all $n \in \mathbb{N}$ and **decreasing** if $a_n \geqslant a_{n+1}$ for all $n \in \mathbb{N}$. A sequence is **monotone** if it is either increasing or decreasing.

Theorem 2.4.1 Theorem 2.4.2 (Monotone Convergence Theorem).
If a sequence is **monotone** and bounded, then it converges.

Definition 2.4.2: Definition 2.4.3 (Convergence of a Series).

Let (b_n) be a sequence. An infinite series is a formal expression of the form

$$\sum_{n=1}^{\infty} b_n = b_1 + b_2 + b_3 + b_4 + b_5 + \cdots$$

We define the corresponding sequence of partial sums (s_m) by

$$s_m = b_1 + b_2 + b_3 + \cdots + b_m$$

and say that the series $\sum_{n=1}^{\infty} b_n$ converges to B if the sequence (s_m) converges to B. In this case, we write $\sum_{n=1}^{\infty} b_n = B$.

Example 2.4.1 (Example 2.4.5 (Harmonic Series).)
This time, consider the so-called harmonic series

$$\sum_{n=1}^{\infty} \frac{1}{n}$$

Again, we have an increasing sequence of partial sums,

$$s_m = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{m}$$

that upon naive inspection appears as though it may be bounded. However, 2 is no longer an upper bound because

$$s_4 = 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) > 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) = 2.$$

A similar calculation shows that $s_8 > 2\frac{1}{2}$, and we can see that in general

$$\begin{aligned}
s_{2^k} &= 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \cdots + \frac{1}{8}\right) + \cdots + \left(\frac{1}{2^{k-1}+1} + \cdots + \frac{1}{2^k}\right) \\
&> 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + \cdots + \frac{1}{8}\right) + \cdots + \left(\frac{1}{2^k} + \cdots + \frac{1}{2^k}\right) \\
&= 1 + \frac{1}{2} + 2\left(\frac{1}{4}\right) + 4\left(\frac{1}{8}\right) + \cdots + 2^{k-1}\left(\frac{1}{2^k}\right) \\
&= 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \cdots + \frac{1}{2} \\
&= 1 + k\left(\frac{1}{2}\right)
\end{aligned}$$

which is unbounded. Thus, despite the incredibly slow pace, the sequence of partial sums of $\sum_{n=1}^{\infty} 1/n$ eventually surpasses every number on the positive real line. Because convergent sequences are bounded, the harmonic series diverges.

Theorem 2.4.2 Theorem 2.4.6 (Cauchy Condensation Test).

Suppose (b_n) is decreasing and satisfies $b_n \geqslant 0$ for all $n \in \mathbb{N}$. Then, the series $\sum_{n=1}^{\infty} b_n$ converges if and only if the series

$$\sum_{n=0}^{\infty} 2^n b_{2^n} = b_1 + 2b_2 + 4b_4 + 8b_8 + 16b_{16} + \cdots$$

converges.

Corollary 2.4.1 Corollary 2.4.7.

The series $\sum_{n=1}^{\infty} 1/n^p$ converges if and only if $p > 1$.

2.5 Subsequences and the Bolzano-Weierstrass Theorem

Definition 2.5.1: Definition 2.5.1.

Let (a_n) be a sequence of real numbers, and let $n_1 < n_2 < n_3 < n_4 < n_5 < \ldots$ be an increasing sequence of natural numbers. Then the sequence

$$(a_{n_1}, a_{n_2}, a_{n_3}, a_{n_4}, a_{n_5}, \ldots)$$

is called a **subsequence** of (a_n) and is denoted by $\left(a_{n_k}\right)$, where $k \in \mathbb{N}$ indexes the subsequence.

Theorem 2.5.1 Theorem 2.5.2.

Subsequences of a convergent sequence converge to the same limit as the original sequence.

Theorem 2.5.2 Theorem 2.5.5 (Bolzano-Weierstrass Theorem).

Every bounded sequence contains a convergent subsequence.

2.6 The Cauchy Criterion

Definition 2.6.1: Definition 2.6.1.

A sequence (a_n) is called a Cauchy sequence if, for every $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that whenever $m, n \geqslant N$ it follows that $|a_n - a_m| < \epsilon$

Theorem 2.6.1 Theorem 2.6.2.

Every convergent sequence is a Cauchy sequence.

Lemma 2.6.1 Lemma 2.6.3.

Cauchy sequences are bounded.

Theorem 2.6.2 Theorem 2.6.4 (Cauchy Criterion).
A sequence converges if and only if it is a Cauchy sequence.

2.7 Properties of Infinite Series

Given an infinite series $\sum_{k=1}^{\infty} a_k$, it is important to keep a clear distinction between

(a) the sequence of terms: $(a_1, a_2, a_3, \ldots)$ and

(b) the sequence of partial sums: $(s_1, s_2, s_3, \ldots)$, where $s_n = a_1 + a_2 + \cdots + a_n$.

The convergence of the series $\sum_{k=1}^{\infty} a_k$ is defined in terms of the sequence (s_n). Specifically, the statement

$$\sum_{k=1}^{\infty} a_k = A \quad \text{means that} \quad \lim s_n = A.$$

Theorem 2.7.1 Theorem 2.7.1 (Algebraic Limit Theorem for Series).
If $\sum_{k=1}^{\infty} a_k = A$ and $\sum_{k=1}^{\infty} b_k = B$, then

(a) $\sum_{k=1}^{\infty} ca_k = cA$ for all $c \in \mathbb{R}$ and

(b) $\sum_{k=1}^{\infty} (a_k + b_k) = A + B$.

Theorem 2.7.2 Theorem 2.7.2 (Cauchy Criterion for Series).
The series $\sum_{k=1}^{\infty} a_k$ converges if and only if, given $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that whenever $n > m \geqslant N$ it follows that

$$|a_{m+1} + a_{m+2} + \cdots + a_n| < \epsilon$$

Theorem 2.7.3 Theorem 2.7.3.
If the series $\sum_{k=1}^{\infty} a_k$ converges, then $(a_k) \to 0$.

Theorem 2.7.4 Theorem 2.7.4 (Comparison Test).
Assume (a_k) and (b_k) are sequences satisfying $0 \leqslant a_k \leqslant b_k$ for all $k \in \mathbb{N}$.

(a) If $\sum_{k=1}^{\infty} b_k$ converges, then $\sum_{k=1}^{\infty} a_k$ converges.

(b) If $\sum_{k=1}^{\infty} a_k$ diverges, then $\sum_{k=1}^{\infty} b_k$ diverges.

Example 2.7.1 (Example 2.7.5 (Geometric Series))
A series is called geometric if it is of the form

$$\sum_{k=0}^{\infty} ar^k = a + ar + ar^2 + ar^3 + \cdots$$

If $r = 1$ and $a \neq 0$, the series evidently diverges. For $r \neq 1$, the algebraic identity

$$(1 - r)\left(1 + r + r^2 + r^3 + \cdots + r^{m-1}\right) = 1 - r^m$$

enables us to rewrite the partial sum

$$s_m = a + ar + ar^2 + ar^3 + \cdots + ar^{m-1} = \frac{a(1 - r^m)}{1 - r}$$

Now the Algebraic Limit Theorem (for sequences) and Example 2.5.3 justify the conclusion

$$\sum_{k=0}^{\infty} ar^k = \frac{a}{1-r}$$

if and only if $|r| < 1$.

Theorem 2.7.5 Theorem 2.7.6 (Absolute Convergence Test).
If the series $\sum_{n=1}^{\infty} |a_n|$ converges, then $\sum_{n=1}^{\infty} a_n$ converges as well.

Theorem 2.7.6 Theorem 2.7.7 (Alternating Series Test).
Let (a_n) be a sequence satisfying,

(a) $a_1 \geqslant a_2 \geqslant a_3 \geqslant \cdots \geqslant a_n \geqslant a_{n+1} \geqslant \cdots$ and

(b) $(a_n) \to 0$.

Then, the alternating series $\sum_{n=1}^{\infty} (-1)^{n+1} a_n$ converges.

Definition 2.7.1: Definition 2.7.8.

If $\sum_{n=1}^{\infty} |a_n|$ converges, then we say that the original series $\sum_{n=1}^{\infty} a_n$ **converges absolutely**. If, on the other hand, the series $\sum_{n=1}^{\infty} a_n$ converges but the series of absolute values $\sum_{n=1}^{\infty} |a_n|$ does not converge, then we say that the original series $\sum_{n=1}^{\infty} a_n$ **converges conditionally.**

Note:-
In terms of this newly defined jargon, we have shown that

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$$

converges conditionally, whereas

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}, \quad \sum_{n=1}^{\infty} \frac{1}{2^n} \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2^n}$$

converge absolutely.

2.7.1 Rearrangements

Definition 2.7.2: Definition 2.7.9.

Let $\sum_{k=1}^{\infty} a_k$ be a series. A series $\sum_{k=1}^{\infty} b_k$ is called a **rearrangement** of $\sum_{k=1}^{\infty} a_k$ if there exists a one-to-one, onto function $f : \mathbb{N} \to \mathbb{N}$ such that $b_{f(k)} = a_k$ for all $k \in \mathbb{N}$.

Theorem 2.7.7 Theorem 2.7.10.
If a series converges absolutely, then any rearrangeent of this series converges to the same limit.

Chapter 3

Basic Topology of R

3.1 Discussion: The Cantor Set

TLDR: it's pretty cool.

3.2 Open and Closed Sets

Given $a \in \mathbf{R}$ and $\epsilon > 0$, recall that the ϵ-neighborhood of a is the set

$$V_\epsilon(a) = \{x \in \mathbf{R} : |x - a| < \epsilon\}$$

In other words, $V_\epsilon(a)$ is the open interval $(a - \epsilon, a + \epsilon)$, centered at a with radius ϵ.

Definition 3.2.1: Definition 3.2.1.

A set $O \subseteq \mathbf{R}$ is **open** if for all points $a \in O$ there exists an ϵ-neighborhood $V_\epsilon(a) \subseteq O$.

Theorem 3.2.1 Theorem 3.2.3.

(a) The union of an arbitrary collection of open sets is open.

(b) The intersection of a finite collection of open sets is open.

3.2.1 Closed Sets

Definition 3.2.2: Definition 3.2.4.

A point x is a **limit point** of a set A if every ϵ-neighborhood $V_\epsilon(x)$ of x intersects the set A at some point other than x.

Theorem 3.2.2 Theorem 3.2.5.

A point x is a limit point of a set A if and only if $x = \lim a_n$ for some sequence (a_n) contained in A satisfying $a_n \neq x$ for all $n \in \mathbf{N}$.

Definition 3.2.3: Definition 3.2.6.

A point $a \in A$ is an **isolated point** of A if it is not a limit point of A.

Definition 3.2.4: Definition 3.2.7.

A set $F \subseteq \mathbf{R}$ is **closed** if it contains its limit points.

Theorem 3.2.3 Theorem 3.2.8.

A set $F \subseteq \mathbf{R}$ is closed if and only if every Cauchy sequence contained in F has a limit that is also an element of F.

Example 3.2.1 (Example 3.2.9.)

(a) Consider

$$A = \left\{ \frac{1}{n} : n \in \mathbf{N} \right\}$$

Let's show that each point of A is isolated. Given $1/n \in A$, choose $\epsilon = 1/n - 1/(n+1)$. Then,

$$V_\epsilon(1/n) \cap A = \left\{ \frac{1}{n} \right\}$$

It follows from Definition 3.2.4 that $1/n$ is not a limit point and so is isolated. Although all of the points of A are isolated, the set does have one limit point, namely 0 . This is because every neighborhood centered at zero, no matter how small, is going to contain points of A. Because $0 \notin A, A$ is not closed. The set $F = A \cup \{0\}$ is an example of a closed set and is called the closure of A. (The closure of a set is discussed in a moment.)

(b) Let's prove that a closed interval

$$[c, d] = \{x \in \mathbf{R} : c \leqslant x \leqslant d\}$$

is a closed set using Definition 3.2.7. If x is a limit point of $[c, d]$, then by Theorem 3.2.5 there exists $(x_n) \subseteq [c, d]$ with $(x_n) \to x$. We need to prove that $x \in [c, d]$. The key to this argument is contained in the Order Limit Theorem (Theorem 2.3.4), which summarizes the relationship between inequalities and the limiting process. Because $c \leqslant x_n \leqslant d$, it follows from Theorem 2.3.4 (iii) that $c \leqslant x \leqslant d$ as well. Thus, $[c, d]$ is closed.

(c) Consider the set $\mathbf{Q} \subseteq \mathbf{R}$ of rational numbers. An extremely important property of $\mathbf{Q}$ is that its set of limit points is actually all of $\mathbf{R}$. To see why this is so, recall Theorem 1.4.3 from Chapter 1, which is referred to as the density property of $\mathbf{Q}$ in $\mathbf{R}$. Let $y \in \mathbf{R}$ be arbitrary, and consider any neighborhood $V_\epsilon(y) = (y - \epsilon,\, y + \epsilon)$. Theorem 1.4.3 allows us to conclude that there exists a rational number $r \neq y$ that falls in this neighborhood. Thus, y is a limit point of $\mathbf{Q}$.

Theorem 3.2.4 Theorem 3.2.10 (Density of $\mathbf{Q}$ in $\mathbf{R}$)

For every $y \in \mathbf{R}$, there exists a sequence of rational numbers that converges to y.

3.2.2 Closure

Definition 3.2.5: Definition 3.2.11.

Given a set $A \subseteq \mathbf{R}$, let L be the set of all limit points of A. The **closure** of A is defined to be $\bar{A} = A \cup L$.

Theorem 3.2.5 Theorem 3.2.12.

For any $A \subseteq \mathbf{R}$, the closure $\bar{A}$ is a closed set and is the smallest closed set containing A.

3.2.3 Complements

Recall that the **complement** of a set $A \subseteq \mathbb{R}$ is defined to be the set

$$A^c = \{x \in \mathbf{R} : x \notin A\}.$$

Theorem 3.2.6 Theorem 3.2.13.
A set O is open if and only if O^c is closed. Likewise, a set F is closed if and only if F^c is open.

Theorem 3.2.7 Theorem 3.2.14.

(a) The union of a finite collection of closed sets is closed.

(b) The intersection of an arbitrary collection of closed sets is closed.

3.3 Compact Sets

Definition 3.3.1: Definition 3.3.1 (Compactness).

A set $K \subseteq \mathbf{R}$ is **compact** if every sequence in K has a subsequence that converges to a limit that is also in K.

Definition 3.3.2: Definition 3.3.3.

A set $A \subseteq \mathbf{R}$ is **bounded** if there exists $M > 0$ such that $|a| \leqslant M$ for all $a \in A$.

Theorem 3.3.1 Theorem 3.3.4 (Characterization of Compactness in $\mathbf{R}$)
A set $K \subseteq \mathbf{R}$ is compact if and only if it is closed and bounded.

Theorem 3.3.2 Theorem 3.3.5 (Nested Compact Set Property).
If

$$K_1 \supseteq K_2 \supseteq K_3 \supseteq K_4 \supseteq \cdots$$

is a nested sequence of nonempty compact sets, then the intersection $\bigcap_{n=1}^{\infty} K_n$ is not empty.

3.3.1 Open Covers

Definition 3.3.3: Definition 3.3.6.

Let $A \subseteq \mathbf{R}$. An **open cover** for A is a (possibly infinite) collection of open sets $\{O_\lambda : \lambda \in \Lambda\}$ whose union contains the set A; that is, $A \subseteq \bigcup_{\lambda \in \Lambda} O_\lambda$. Given an open cover for A, a **finite subcover** is a finite subcollection of open sets from the original open cover whose union still manages to completely contain A.

Theorem 3.3.3 Theorem 3.3.8 (Heine-Borel Theorem).
Let K be a subset of R. All of the following statements are equivalent in the sense that any one of them implies the two others:

(a) K is compact.

(b) K is closed and bounded.

(c) Every open cover for K has a finite subcover.

3.4 Perfect Sets and Connected Sets

3.4.1 Perfect Sets

Definition 3.4.1: Definition 3.4.1.

A set $P \subseteq \mathbf{R}$ is **perfect** if it is closed and contains no isolated points.

Example 3.4.1

Closed intervals (other than the singleton sets $[a, a]$) serve as the most obvious class of perfect sets, but there are more interesting examples.

Example 3.4.2 (Example 3.4.2 (Cantor Set))

It is not too hard to see that the Cantor set is perfect. In Section 3.1, we defined the Cantor set as the intersection

$$C = \bigcap_{n=0}^{\infty} C_n$$

where each C_n is a finite union of closed intervals. By Theorem 3.2.14, each C_n is closed, and by the same theorem, C is closed as well. It remains to show that no point in C is isolated. Let $x \in C$ be arbitrary. To convince ourselves that x is not isolated, we must construct a sequence (x_n) of points in C, different from x, that converges to x. From our earlier discussion, we know that C at least contains the endpoints of the intervals that make up each C_n. In Exercise 3.4.3, we sketch the argument that these are all that is needed to construct (x_n).

Theorem 3.4.1 Theorem 3.4.3.

A nonempty perfect set is uncountable.

3.4.2 Connected Sets

Definition 3.4.2: Definition 3.4.4.

Two nonempty sets $A, B \subseteq \mathbf{R}$ are **separated** if $\bar{A} \cap B$ and $A \cap \bar{B}$ are both empty. A set $E \subseteq \mathbf{R}$ is **disconnected** if it can be written as $E = A \cup B$, where A and B are nonempty separated sets. A set that is not disconnected is called a **connected** set.

Theorem 3.4.2 Theorem 3.4.6.

A set $E \subseteq \mathbf{R}$ is connected if and only if, for all nonempty disjoint sets A and B satisfying $E = A \cup B$, there always exists a convergent sequence $(x_n) \to x$ with (x_n) contained in one of A or B, and x an element of the other.

Theorem 3.4.3 Theorem 3.4.7.

A set $E \subseteq \mathbf{R}$ is connected if and only if whenever $a < c < b$ with $a, b \in E$, it follows that $c \in E$ as well.

Chapter 4

Functional Limits and Continuity

4.1 Discussion: Examples of Dirichlet and Thomae

Dirichlet's Function:

$$g(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q}. \end{cases}$$

Dirichlet's function is discontinuous on all of $\mathbb{R}$. Since the rationals and irrationals are dense in the reals, we can always pick a sequence of rationals and a sequence of irrationals that approach $c \in \mathbb{R}$. However, by construction of Dirichlet's function, these sequences necessarily converge to different limits.

We can make a modification to Dirichlet's Function:

$$h(x) = \begin{cases} x & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q}. \end{cases}$$

By the same argument as above, we find h is discontinuous at all points except $x = 0$.

Thomae's Function:

$$t(x) = \begin{cases} 1 & \text{if } x = 0 \\ 1/n & \text{if } x = m/n \in \mathbb{Q}\backslash\{0\} \text{ is in lowest terms with } n > 0 \\ 0 & \text{if } x \notin \mathbb{Q}. \end{cases}$$

Thomae's function is discontinuous over all of $\mathbb{Q}$. If $c \in \mathbb{Q}$, we can find a sequence $(y_n) \subseteq \mathbb{I}$ converging to c. Then, $\lim t(y_n) = 0 \neq t(c)$.

4.2 Functional Limits

Consider a function $f : A \to \mathbf{R}$. Recall that a limit point c of A is a point with the property that every ϵ-neighborhood $V_\epsilon(c)$ intersects A in some point other than c. Equivalently, c is a limit point of A if and only if $c = \lim x_n$ for some sequence $(x_n) \subseteq A$ with $x_n \neq c$. It is important to remember that limit points of A do not necessarily belong to the set A unless A is closed.

Definition 4.2.1: Definition 4.2.1 (Functional Limit)

Let $f : A \to \mathbf{R}$, and let c be a limit point of the domain A. We say that $\lim_{x\to c} f(x) = L$ provided that, for all $\epsilon > 0$, there exists a $\delta > 0$ such that whenever $0 < |x - c| < \delta$ (and $x \in A$) it follows that $|f(x) - L| < \epsilon$.

Definition 4.2.2: Definition 4.2.1B (Functional Limit: Topological Version).

Let c be a limit point of the domain of $f : A \to \mathbf{R}$. We say $\lim_{x\to c} f(x) = L$ provided that, for every ϵ-neighborhood $V_\epsilon(L)$ of L, there exists a δ-neighborhood $V_\delta(c)$ around c with the property that for all $x \in V_\delta(c)$ different from c (with $x \in A$) it follows that $f(x) \in V_\epsilon(L)$.

Example 4.2.1 (Example 4.2.2.)

(a) To familiarize ourselves with Definition 4.2.1, let's prove that if $f(x) = 3x + 1$, then

$$\lim_{x \to 2} f(x) = 7$$

Let $\epsilon > 0$. Definition 4.2 .1 requires that we produce a $\delta > 0$ so that $0 < |x - 2| < \delta$ leads to the conclusion $|f(x) - 7| < \epsilon$. Notice that

$$|f(x) - 7| = |(3x + 1) - 7| = |3x - 6| = 3|x - 2|$$

Thus, if we choose $\delta = \epsilon/3$, then $0 < |x - 2| < \delta$ implies $|f(x) - 7| < 3(\epsilon/3) = \epsilon$.

(b) Let's show

$$\lim_{x \to 2} g(x) = 4$$

where $g(x) = x^2$. Given an arbitrary $\epsilon > 0$, our goal this time is to make $|g(x) - 4| < \epsilon$ by restricting $|x - 2|$ to be smaller than some carefully chosen δ. As in the previous problem, a little algebra reveals

$$|g(x) - 4| = \left|x^2 - 4\right| = |x + 2||x - 2|$$

We can make $|x - 2|$ as small as we like, but we need an upper bound on $|x + 2|$ in order to know how small to choose δ. The presence of the variable x causes some initial confusion, but keep in mind that we are discussing the limit as x approaches 2 . If we agree that our δ-neighborhood around $c = 2$ must have radius no bigger than $\delta = 1$, then we get the upper bound $|x + 2| \leqslant |3 + 2| = 5$ for all $x \in V_\delta(c)$. Now, choose $\delta = \min\{1, \epsilon/5\}$. If $0 < |x - 2| < \delta$, then it follows that

$$\left|x^2 - 4\right| = |x + 2||x - 2| < (5)\frac{\epsilon}{5} = \epsilon$$

and the limit is proved.

4.2.1 Sequential Criterion for Functional Limits

Theorem 4.2.1 Theorem 4.2.3 (Sequential Criterion for Functional Limits).

Given a function $f : A \to \mathbf{R}$ and a limit point c of A, the following two statements are equivalent:

(a) $\lim_{x \to c} f(x) = L$.

(b) For all sequences $(x_n) \subseteq A$ satisfying $x_n \neq c$ and $(x_n) \to c$, it follows that $f(x_n) \to L$.

Corollary 4.2.1 Corollary 4.2.4 (Algebraic Limit Theorem for Functional Limits).

Let f and g be functions defined on a domain $A \subseteq \mathbf{R}$, and assume $\lim_{x \to c} f(x) = L$ and $\lim_{x \to c} g(x) = M$ for some limit point c of A. Then,

(a) $\lim_{x \to c} kf(x) = kL$ for all $k \in \mathbf{R}$,

(b) $\lim_{x \to c}[f(x) + g(x)] = L + M$,

(c) $\lim_{x \to c}[f(x)g(x)] = LM$, and

(d) $\lim_{x \to c} f(x)/g(x) = L/M$, provided $M \neq 0$.

Corollary 4.2.2 Corollary 4.2.5 (Divergence Criterion for Functional Limits).

Let f be a function defined on A, and let c be a limit point of A. If there exist two sequences (x_n) and (y_n) in A with $x_n \neq c$ and $y_n \neq c$ and

$$\lim x_n = \lim y_n = c \quad \text{but} \quad \lim f(x_n) \neq \lim f(y_n)$$

then we can conclude that the functional limit $\lim_{x \to c} f(x)$ does not exist.

Example 4.2.2 (Example 4.2.6.)

Assuming the familiar properties of the sine function, let's show that $\lim_{x \to 0} \sin(1/x)$ does not exist (Fig. 4.5). If $x_n = 1/2n\pi$ and $y_n = 1/(2n\pi + \pi/2)$, then $\lim(x_n) = \lim(y_n) = 0$. However, $\sin(1/x_n) = 0$ for all $n \in \mathbf{N}$ while $\sin(1/y_n) = 1$. Thus,

$$\lim \sin(1/x_n) \neq \lim \sin(1/y_n)$$

so by Corollary 4.2.5, $\lim_{x \to 0} \sin(1/x)$ does not exist.

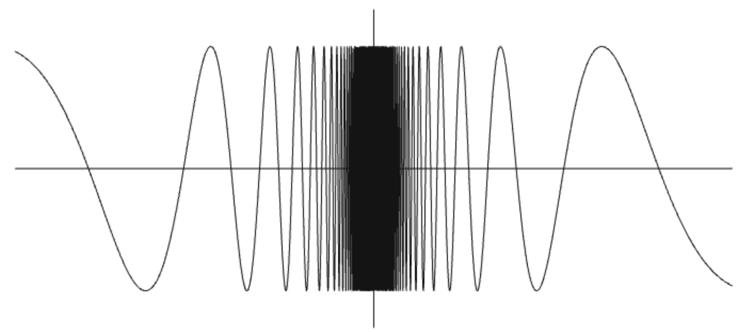

Figure 4.5: The function $\sin(1/x)$ near zero.

4.3 Continuous Functions

Definition 4.3.1: Definition 4.3.1 (Continuity).

A function $f : A \to \mathbf{R}$ is **continuous** at a point $c \in A$ if, for all $\epsilon > 0$, there exists a $\delta > 0$ such that whenever $|x - c| < \delta$ (and $x \in A$) it follows that $|f(x) - f(c)| < \epsilon$. If f is continuous at every point in the domain A, then we say that f is **continuous** on A.

Theorem 4.3.1 Theorem 4.3.2 (Characterizations of Continuity).

Let $f : A \to \mathbf{R}$, and let $c \in A$. The function f is continuous at c if and only if any one of the following three conditions is met:

(a) For all $\epsilon > 0$, there exists a $\delta > 0$ such that $|x - c| < \delta($ and $x \in A)$ implies $|f(x) - f(c)| < \epsilon$;

(b) For all $V_\epsilon(f(c))$, there exists a $V_\delta(c)$ with the property that $x \in V_\delta(c)$ (and $x \in A$) implies $f(x) \in V_\epsilon(f(c))$;

(c) If $(x_n) \to c$ (with $x_n \in A$), then $f(x_n) \to f(c)$.

(d) If c is a limit point of A, then the above conditions are equivalent to

$\lim_{x \to c} f(x) = f(c)$

Corollary 4.3.1 Corollary 4.3.3 (Criterion for Discontinuity).

Let $f : A \to \mathbf{R}$, and let $c \in A$ be a limit point of A. If there exists a sequence $(x_n) \subseteq A$ where $(x_n) \to c$ but such that $f(x_n)$ does not converge to $f(c)$, we may conclude that f is not continuous at c.

Theorem 4.3.2 Theorem 4.3.4 (Algebraic Continuity Theorem).

Assume $f : A \to \mathbf{R}$ and $g : A \to \mathbf{R}$ are continuous at a point $c \in A$. Then,

(a) $kf(x)$ is continuous at c for all $k \in \mathbf{R}$;

(b) $f(x) + g(x)$ is continuous at c;

(c) $f(x)g(x)$ is continuous at c; and

(d) $f(x)/g(x)$ is continuous at c, provided the quotient is defined.

Example 4.3.1 (Example 4.3.5.)

All polynomials are continuous on $\mathbf{R}$. In fact, rational functions (i.e., quotients of polynomials) are continuous wherever they are defined.

To see why this is so, consider the identity function $g(x) = x$. Because $|g(x) - g(c)| = |x - c|$, we can respond to a given $\epsilon > 0$ by choosing $\delta = \epsilon$, and it follows that g is continuous on all of $\mathbf{R}$. It is even simpler to show that a constant function $f(x) = k$, is continuous. (Letting $\delta = 1$ regardless of the value of ϵ does the trick.) Because an arbitrary polynomial

$$p(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$$

consists of sums and products of $g(x)$ with different constant functions, we may conclude from Theorem 4.3.4 that $p(x)$ is continuous.

Likewise, Theorem 4.3.4 implies that quotients of polynomials are continuous as long as the denominator is not zero.

Example 4.3.2 (Example 4.3.8.)

Consider $f(x) = \sqrt{x}$ defined on $A = \{x \in \mathbf{R} : x \geqslant 0\}$. Exercise 2.3.1 outlines a sequential proof that f is continuous on A. Here, we give an $\epsilon - \delta$ proof of the same fact.

Let $\epsilon > 0$. We need to argue that $|f(x) - f(c)|$ can be made less than ϵ for all values of x in some δ neighborhood around c. If $c = 0$, this reduces to the statement $\sqrt{x} < \epsilon$, which happens as long as $x < \epsilon^2$. Thus, if we choose $\delta = \epsilon^2$, we see that $|x - 0| < \delta$ implies $|f(x) - 0| < \epsilon$.

For a point $c \in A$ different from zero, we need to estimate $|\sqrt{x} - \sqrt{c}|$. This time, write

$$|\sqrt{x} - \sqrt{c}| = |\sqrt{x} - \sqrt{c}| \left(\frac{\sqrt{x} + \sqrt{c}}{\sqrt{x} + \sqrt{c}} \right) = \frac{|x - c|}{\sqrt{x} + \sqrt{c}} \leqslant \frac{|x - c|}{\sqrt{c}}.$$

In order to make this quantity less than ϵ, it suffices to pick $\delta = \epsilon\sqrt{c}$. Then, $|x - c| < \delta$ implies

$$|\sqrt{x} - \sqrt{c}| < \frac{\epsilon\sqrt{c}}{\sqrt{c}} = \epsilon$$

as desired.

Theorem 4.3.3 Theorem 4.3.9 (Composition of Continuous Functions).

Given $f : A \to \mathbf{R}$ and $g : B \to \mathbf{R}$, assume that the range $f(A) = \{f(x) : x \in A\}$ is contained in the domain B so that the composition $g \circ f(x) = g(f(x))$ is defined on A.

If f is continuous at $c \in A$, and if g is continuous at $f(c) \in B$, then $g \circ f$ is continuous at c.

4.4 Continuous Functions on Compact Sets

Given a function $f : A \to \mathbf{R}$ and a subset $B \subseteq A$, the notation $f(B)$ refers to the range of f over the set B; that is,

$$f(B) = \{f(x) : x \in B\}$$

Theorem 4.4.1 Theorem 4.4.1 (Preservation of Compact Sets).
Let $f : A \to \mathbf{R}$ be continuous on A. If $K \subseteq A$ is compact, then $f(K)$ is compact as well.

Theorem 4.4.2 Theorem 4.4.2 (Extreme Value Theorem).
If $f : K \to \mathbf{R}$ is continuous on a compact set $K \subseteq \mathbf{R}$, then f attains a maximum and minimum value. In other words, there exist $x_0, x_1 \in K$ such that $f(x_0) \leqslant f(x) \leqslant f(x_1)$ for all $x \in K$.

4.4.1 Uniform Continuity

Example 4.4.1 (Example 4.4.3.)

(a) To show directly that $f(x) = 3x + 1$ is continuous at an arbitrary point $c \in \mathbf{R}$, we must argue that $|f(x) - f(c)|$ can be made arbitrarily small for values of x near c. Now,

$$|f(x) - f(c)| = |(3x + 1) - (3c + 1)| = 3|x - c|$$

so, given $\epsilon > 0$, we choose $\delta = \epsilon/3$. Then, $|x - c| < \delta$ implies

$$|f(x) - f(c)| = 3|x - c| < 3\left(\frac{\epsilon}{3}\right) = \epsilon$$

Of particular importance for this discussion is the fact that the choice of δ is the same regardless of which point $c \in \mathbf{R}$ we are considering.

(b) Let's contrast this with what happens when we prove $g(x) = x^2$ is continuous on $\mathbf{R}$. Given $c \in \mathbf{R}$, we have

$$|g(x) - g(c)| = \left|x^2 - c^2\right| = |x - c||x + c|$$

As discussed in Example 4.2.2, we need an upper bound on $|x + c|$, which is obtained by insisting that our choice of δ not exceed 1 . This guarantees that all values of x under consideration will necessarily fall in the interval $(c - 1, c + 1)$. It follows that

$$|x + c| \leqslant |x| + |c| \leqslant (|c| + 1) + |c| = 2|c| + 1$$

Now, let $\epsilon > 0$. If we choose $\delta = \min\{1, \epsilon/(2|c| + 1)\}$, then $|x - c| < \delta$ implies

$$|f(x) - f(c)| = |x - c||x + c| < \left(\frac{\epsilon}{2|c| + 1}\right)(2|c| + 1) = \epsilon$$

Definition 4.4.1: Definition 4.4.4 (Uniform Continuity).

A function $f : A \to \mathbf{R}$ is **uniformly continuous** on A if for every $\epsilon > 0$ there exists a $\delta > 0$ such that for all $x, y \in A$, $|x - y| < \delta$ implies $|f(x) - f(y)| < \epsilon$.

Theorem 4.4.3 Theorem 4.4.5 (Sequential Criterion for Absence of Uniform Continuity).

A function $f : A \to \mathbf{R}$ fails to be uniformly continuous on A if and only if there exists a particular $\epsilon_0 > 0$ and two sequences (x_n) and (y_n) in A satisfying

$$|x_n - y_n| \to 0 \quad \text{but} \quad |f(x_n) - f(y_n)| \geqslant \epsilon_0$$

Theorem 4.4.4 Theorem 4.4.7 (Uniform Continuity on Compact Sets).
A function that is continuous on a compact set K is uniformly continuous on K.

4.5 The Intermediate Value Theorem

Theorem 4.5.1 Theorem 4.5.1 (Intermediate Value Theorem).
Let $f : [a, b] \to \mathbf{R}$ be continuous. If L is a real number satisfying $f(a) < L < f(b)$ or $f(a) > L > f(b)$, then there exists a point $c \in (a, b)$ where $f(c) = L$.

Preservation of Connected Sets

Theorem 4.5.2 Theorem 4.5.2 (Preservation of Connected Sets)
Let $f : G \to \mathbf{R}$ be continuous. If $E \subseteq G$ is connected, then $f(E)$ is connected as well.

4.5.1 Completeness

4.5.2 The Intermediate Value Property

Definition 4.5.1: Definition 4.5.3.

A function f has the intermediate value property on an interval $[a, b]$ if for all $x < y$ in $[a, b]$ and all L between $f(x)$ and $f(y)$, it is always possible to find a point $c \in (x, y)$ where $f(c) = L$.

4.6 Sets of Discontinuity

4.6.1 Monotone Functions

Definition 4.6.1: Definition 4.6.1.

A function $f : A \to \mathbf{R}$ is **increasing** on A if $f(x) \leqslant f(y)$ whenever $x < y$ and **decreasing** if $f(x) \geqslant f(y)$ whenever $x < y$ in A. A monotone function is one that is either increasing or decreasing.

Definition 4.6.2: Definition 4.6.2.

Given a limit point c of a set A and a function $f : A \to \mathbf{R}$, we write

$$\lim_{x \to c^+} f(x) = L$$

if for all $\epsilon > 0$ there exists a $\delta > 0$ such that $|f(x) - L| < \epsilon$ whenever $0 < x - c < \delta$. Equivalently, in terms of sequences, $\lim_{x \to c^+} f(x) = L$ if $\lim f(x_n) = L$ for all sequences (x_n) satisfying $x_n > c$ and $\lim(x_n) = c$.

Theorem 4.6.1 Theorem 4.6.3.

Given $f : A \to \mathbf{R}$ and a limit point c of A, $\lim_{x\to c} f(x) = L$ if and only if

$$\lim_{x\to c^-} f(x) = L \quad \text{and} \quad \lim_{x\to c^+} f(x) = L$$

4.6.2 D_f for an Arbitrary Function

Definition 4.6.3: Definition 4.6.4.

A set that can be written as the countable union of closed sets is in the class F_σ. (This definition also appeared in Section 3.5.)

Definition 4.6.4

Definition 4.6.5. Let f be defined on $\mathbf{R}$, and let $\alpha > 0$. The function f is **α-continuous** at $x \in \mathbf{R}$ if there exists a $\delta > 0$ such that for all $y, z \in (x - \delta, x + \delta)$ it follows that $|f(y) - f(z)| < \alpha$.

Theorem 4.6.2 Theorem 4.6.6.

Let $f : \mathbf{R} \to \mathbf{R}$ be an arbitrary function. Then, D_f is an F_σ set.

Chapter 5

The Derivative

5.1 Discussion: Are Derivatives Continuous?

5.2 Derivatives and the Intermediate Value Property

Definition 5.2.1: Definition 5.2.1 (Differentiability).

Let $g : A \to \mathbf{R}$ be a function defined on an interval A. Given $c \in A$, the **derivative** of g at c is defined by

$$g'(c) = \lim_{x \to c} \frac{g(x) - g(c)}{x - c}$$

provided this limit exists. In this case we say g is **differentiable** at c. If g' exists for all points $c \in A$, we say that g is **differentiable** on A.

Theorem 5.2.1 Theorem 5.2.3.

If $g : A \to \mathbf{R}$ is differentiable at a point $c \in A$, then g is continuous at c as well.

5.2.1 Combinations of Differentiable Functions

Theorem 5.2.2 Theorem 5.2.4 (Algebraic Differentiability Theorem)

Let f and g be functions defined on an interval A, and assume both are differentiable at some point $c \in A$. Then,

(a) $(f + g)'(c) = f'(c) + g'(c)$,

(b) $(kf)'(c) = kf'(c)$, for all $k \in \mathbf{R}$,

(c) $(fg)'(c) = f'(c)g(c) + f(c)g'(c)$, and

(d) $(f/g)'(c) = \frac{g(c)f'(c) - f(c)g'(c)}{[g(c)]^2}$, provided that $g(c) \neq 0$.

Theorem 5.2.3 Theorem 5.2.5 (Chain Rule).

Let $f : A \to \mathbf{R}$ and $g : B \to \mathbf{R}$ satisfy $f(A) \subseteq B$ so that the composition $g \circ f$ is defined. If f is differentiable at $c \in A$ and if g is differentiable at $f(c) \in B$, then $g \circ f$ is differentiable at c with $(g \circ f)'(c) = g'(f(c)) \cdot f'(c)$.

Theorem 5.2.4 Theorem 5.2.6 (Interior Extremum Theorem)

Let f be differentiable on an open interval (a, b). If f attains a maximum value at some point $c \in (a, b)$ (i.e., $f(c) \geqslant f(x)$ for all $x \in (a, b)$), then $f'(c) = 0$. The same is true if $f(c)$ is a minimum value.

Theorem 5.2.5 Theorem 5.2.7 (Darboux's Theorem)

If f is differentiable on an interval $[a, b]$, and if α satisfies $f'(a) < \alpha < f'(b)($ or $f'(a) > \alpha > f'(b))$, then there exists a point $c \in (a, b)$ where $f'(c) = \alpha$.

5.3 The Mean Value Theorem

Theorem 5.3.1 Theorem 5.3.1 (Rolle's Theorem)

Let $f : [a, b] \longrightarrow \mathbf{R}$ be continuous on $[a, b]$ and differentiable on (a, b). If $f(a) = f(b)$, then there exists a point $c \in (a, b)$ where $f'(c) = 0$.

Theorem 5.3.2 Theorem 5.3.2 (Mean Value Theorem)

If $f : [a, b] \longrightarrow \mathbf{R}$ is continuous on $[a, b]$ and differentiable on (a, b), then there exists a point $c \in (a, b)$ where

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Corollary 5.3.1 Corollary 5.3.3.

If $g : A \longrightarrow \mathbf{R}$ is differentiable on an interval A and satisfies $g'(x) = 0$ for all $x \in A$, then $g(x) = k$ for some constant $k \in R$.

Corollary 5.3.2 Corollary 5.3.4.

If f and g are differentiable functions on an interval A and satisfy $f'(x) = g'(x)$ for all $x \in A$, then $f(x) = g(x) + k$ for some constant $k \in \mathbf{R}$.

Theorem 5.3.3 Theorem 5.3.5 (Generalized Mean Value Theorem)

If f and g are continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b), then there exists a point $c \in (a, b)$ where

$$[f(b) - f(a)]g'(c) = [g(b) - g(a)]f'(c).$$

If g' is never zero on (a, b), then the conclusion can be stated as

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}$$

5.3.1 L'Hospital's Rules

Theorem 5.3.4 Theorem 5.3.6 (L'Hospital's Rule: 0/0 case).

Let f and g be continuous on an interval containing a, and assume f and g are differentiable on this interval with the possible exception of the point a. If $f(a) = g(a) = 0$ and $g'(x) \neq 0$ for all $x \neq a$, then

$$\lim_{x \to a} \frac{f'(x)}{g'(x)} = L \quad \text{implies} \quad \lim_{x \to a} \frac{f(x)}{g(x)} = L$$

Definition 5.3.1: Definition 5.3.7.

Given $g : A \to \mathbf{R}$ and a limit point c of A, we say that $\lim_{x\to c} g(x) = \infty$ if, for every $M > 0$, there exists a $\delta > 0$ such that whenever $0 < |x - c| < \delta$ it follows that $g(x) \geqslant M$.
We can define $\lim_{x\to c} g(x) = -\infty$ in a similar way.

Theorem 5.3.5 Theorem 5.3.8 (L'Hospital's Rule: ∞/∞ case)

Assume f and g are differentiable on (a, b) and that $g'(x) \neq 0$ for all $x \in (a, b)$. If $\lim_{x\to a} g(x) = \infty$ (or $-\infty$), then

$$\lim_{x\to a} \frac{f'(x)}{g'(x)} = L \quad \text{implies} \quad \lim_{x\to a} \frac{f(x)}{g(x)} = L.$$

Chapter 6

Sequences and Series of Functions

6.1 Discussion: The Power of Power Series

Euler is smart.

6.2 Uniform Convergence of a Sequence of Functions

6.2.1 Pointwise Convergence

Definition 6.2.1: Definition 6.2.1.

For each $n \in \mathbf{N}$, let f_n be a function defined on a set $A \subseteq \mathbf{R}$. The sequence (f_n) of functions converges pointwise on A to a function f if, for all $x \in A$, the sequence of real numbers $f_n(x)$ converges to $f(x)$.
In this case, we write $f_n \rightarrow f, \lim f_n = f$, or $\lim_{n\to\infty} f_n(x) = f(x)$.

6.2.2 Continuity of the Limit Function

6.2.3 Uniform Convergence

Definition 6.2.2: Definition 6.2.3 (Uniform Convergence)

Let (f_n) be a sequence of functions defined on a set $A \subseteq \mathbf{R}$. Then, (f_n) converges uniformly on A to a limit function f defined on A if, for every $\epsilon > 0$, there exists an $N \in \mathbf{N}$ such that $|f_n(x) - f(x)| < \epsilon$ whenever $n \geqslant N$ and $x \in A$.

Definition 6.2.3: Definition 6.2.1B.

Let f_n be a sequence of functions defined on a set $A \subseteq \mathbf{R}$. Then, (f_n) converges pointwise on A to a limit f defined on A if, for every $\epsilon > 0$ and $x \in A$, there exists an $N \in \mathbf{N}$ (perhaps dependent on x) such that $|f_n(x) - f(x)| < \epsilon$ whenever $n \geqslant N$.

6.2.4 Cauchy Criterion

Theorem 6.2.1 Theorem 6.2.5 (Cauchy Criterion for Uniform Convergence)
A sequence of functions (f_n) defined on a set $A \subseteq \mathbf{R}$ converges uniformly on A if and only if for every $\epsilon > 0$ there exists an $N \in \mathbf{N}$ such that $|f_n(x) - f_m(x)| < \epsilon$ whenever $m, n \geqslant N$ and $x \in A$.

6.2.5 Continuity Revisited

Theorem 6.2.2 Theorem 6.2.6 (Continuous Limit Theorem)

Let (f_n) be a sequence of functions defined on $A \subseteq \mathbf{R}$ that converges uniformly on A to a function f. If each f_n is continuous at $c \in A$, then f is continuous at c.

6.3 Uniform Convergence and Differentiation

Theorem 6.3.1 Theorem 6.3.1 (Differentiable Limit Theorem)

Let $f_n \to f$ pointwise on the closed interval $[a, b]$, and assume that each f_n is differentiable. If (f_n') converges uniformly on $[a, b]$ to a function g, then the function f is differentiable and $f' = g$.

Theorem 6.3.2 Theorem 6.3.2.

Let (f_n) be a sequence of differentiable functions defined on the closed interval $[a, b]$, and assume (f_n') converges uniformly on $[a, b]$. If there exists a point $x_0 \in [a, b]$ where $f_n(x_0)$ is convergent, then (f_n) converges uniformly on $[a, b]$.

Theorem 6.3.3 Theorem 6.3.3.

Let (f_n) be a sequence of differentiable functions defined on the closed interval $[a, b]$, and assume (f_n') converges uniformly to a function g on $[a, b]$. If there exists a point $x_0 \in [a, b]$ for which $f_n(x_0)$ is convergent, then (f_n) converges uniformly. Moreover, the limit function $f = \lim f_n$ is differentiable and satisfies $f' = g$.

6.4 Series of Functions

Definition 6.4.1: Definition 6.4.1.

For each $n \in \mathbf{N}$, let f_n and f be functions defined on a set $A \subseteq \mathbf{R}$. The infinite series

$$\sum_{n=1}^{\infty} f_n(x) = f_1(x) + f_2(x) + f_3(x) + \cdots$$

converges pointwise on A to $f(x)$ if the sequence $s_k(x)$ of partial sums defined by

$$s_k(x) = f_1(x) + f_2(x) + \cdots + f_k(x)$$

converges pointwise to $f(x)$.
The series **converges uniformly** on A to f if the sequence $s_k(x)$ converges uniformly on A to $f(x)$.
In either case, we write $f = \sum_{n=1}^{\infty} f_n$ or $f(x) = \sum_{n=1}^{\infty} f_n(x)$, always being explicit about the type of convergence involved.

Theorem 6.4.1 Theorem 6.4.2 (Term-by-term Continuity Theorem)

Let f_n be continuous functions defined on a set $A \subseteq \mathbf{R}$, and assume $\sum_{n=1}^{\infty} f_n$ converges uniformly on A to a function f. Then, f is continuous on A.

Theorem 6.4.2 Theorem 6.4.3 (Term-by-term Differentiability Theorem)

Let f_n be differentiable functions defined on an interval A, and assume $\sum_{n=1}^{\infty} f_n'(x)$ converges uniformly to a limit $g(x)$ on A. If there exists a point $x_0 \in [a, b]$ where $\sum_{n=1}^{\infty} f_n(x_0)$ converges, then the series $\sum_{n=1}^{\infty} f_n(x)$

converges uniformly to a differentiable function $f(x)$ satisfying $f'(x) = g(x)$ on A. In other words,

$$f(x) = \sum_{n=1}^{\infty} f_n(x) \quad \text{and} \quad f'(x) = \sum_{n=1}^{\infty} f_n'(x).$$

Theorem 6.4.3 Theorem 6.4.4 (Cauchy Criterion for Uniform Convergence of Series)

A series $\sum_{n=1}^{\infty} f_n$ converges uniformly on $A \subseteq \mathbf{R}$ if and only if for every $\epsilon > 0$ there exists an $N \in \mathbf{N}$ such that

$$|f_{m+1}(x) + f_{m+2}(x) + f_{m+3}(x) + \cdots + f_n(x)| < \epsilon$$

whenever $n > m \geqslant N$ and $x \in A$.

Corollary 6.4.1 Corollary 6.4.5 (Weierstrass M-Test)

For each $n \in \mathbf{N}$, let f_n be a function defined on a set $A \subseteq \mathbf{R}$, and let $M_n > 0$ be a real number satisfying

$$|f_n(x)| \leqslant M_n$$

for all $x \in A$. If $\sum_{n=1}^{\infty} M_n$ converges, then $\sum_{n=1}^{\infty} f_n$ converges uniformly on A.

6.5 Power Series

Theorem 6.5.1 Theorem 6.5.1.

If a power series $\sum_{n=0}^{\infty} a_n x^n$ converges at some point $x_0 \in \mathbf{R}$, then it converges absolutely for any x satisfying $|x| < |x_0|$.

6.5.1 Establishing Uniform Convergence

Theorem 6.5.2 Theorem 6.5.2.

If a power series $\sum_{n=0}^{\infty} a_n x^n$ converges absolutely at a point x_0, then it converges uniformly on the closed interval $[-c, c]$, where $c = |x_0|$.

6.5.2 Abel's Theorem

We should remark that if the power series $g(x) = \sum_{n=0}^{\infty} a_n x^n$ converges conditionally at $x = R$, then it is possible for it to diverge when $x = -R$. The series

$$\sum_{n=1}^{\infty} \frac{(-1)^n x^n}{n}$$

with $R = 1$ is an example.

Lemma 6.5.1 Lemma 6.5.3 (Abel's Lemma)

Let b_n satisfy $b_1 \geqslant b_2 \geqslant b_3 \geqslant \cdots \geqslant 0$, and let $\sum_{n=1}^{\infty} a_n$ be a series for which the partial sums are bounded. In other words, assume there exists $A > 0$ such that

$$|a_1 + a_2 + \cdots + a_n| \leqslant A$$

for all $n \in N$. Then, for all $n \in \mathbf{N}$,

$$|a_1 b_1 + a_2 b_2 + a_3 b_3 + \cdots + a_n b_n| \leqslant A b_1.$$

Theorem 6.5.3 Theorem 6.5.4 (Abel's Theorem)

Let $g(x) = \sum_{n=0}^{\infty} a_n x^n$ be a power series that converges at the point $x = R > 0$. Then the series converges uniformly on the interval $[0, R]$. A similar result holds if the series converges at $x = -R$.

6.5.3 The Success of Power Series

Theorem 6.5.4 Theorem 6.5.5.

If a power series converges pointwise on the set $A \subseteq \mathbf{R}$, then it converges uniformly on any compact set $K \subseteq A$.

Theorem 6.5.5 Theorem 6.5.6.

If $\sum_{n=0}^{\infty} a_n x^n$ converges for all $x \in (-R, R)$, then the differentiated series $\sum_{n=1}^{\infty} n a_n x^{n-1}$ converges at each $x \in (-R, R)$ as well. Consequently, the convergence is uniform on compact sets contained in $(-R, R)$.

Theorem 6.5.6 Theorem 6.5.7.

Assume

$$f(x) = \sum_{n=0}^{\infty} a_n x^n$$

converges on an interval $A \subseteq \mathbf{R}$. The function f is continuous on A and differentiable on any open interval $(-R, R) \subseteq A$. The derivative is given by

$$f'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}.$$

Moreover, f is infinitely differentiable on $(-R, R)$, and the successive derivatives can be obtained via term-by-term differentiation of the appropriate series.

6.6 Taylor Series

6.6.1 Manipulating Series

6.6.2 Taylor's Formulas for the Coefficients

Theorem 6.6.1 Theorem 6.6.2 (Taylor's Formula)

Let

$$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + \cdots$$

be defined on some nontrivial interval centered at zero. Then,

$$a_n = \frac{f^{(n)}(0)}{n!}.$$

6.6.3 Lagrange's Remainder Theorem

Theorem 6.6.2 Theorem 6.6.3 (Lagrange's Remainder Theorem)

Let f be differentiable $N + 1$ times on $(-R, R)$, define $a_n = f^{(n)}(0)/n!$ for $n = 0, 1, \ldots, N$, and let

$$S_N(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_N x^N.$$

Given $x \neq 0$ in $(-R, R)$, there exists a point c satisfying $|c| < |x|$ where the error function $E_N(x) =$

$f(x) - S_N(x)$ satisfies

$$E_N(x) = \frac{f^{(N+1)}(c)}{(N+1)!} x^{N+1}$$

6.6.4 Taylor Series Centered at $a \neq 0$

Throughout this chapter we have focused our attention on series expansions centered at zero, but there is nothing special about zero other than notational simplicity. If f is defined in some neighborhood of $a \in \mathbf{R}$ and infinitely differentiable at a, then the Taylor series expansion around a takes the form

$$\sum_{n=0}^{\infty} c_n (x-a)^n \quad \text{where} \quad c_n = \frac{f^{(n)}(a)}{n!}$$

Setting $E_N(x) = f(x) - S_N(x)$ as usual, Lagrange's Remainder Theorem in this case says that there exists a value c between a and x where

$$E_N(x) = \frac{f^{(N+1)}(c)}{(N+1)!} (x-a)^{N+1}$$

6.6.5 A Counterexample

Let

$$g(x) = \begin{cases} e^{-1/x^2} & \text{for } x \neq 0 \\ 0 & \text{for } x = 0 \end{cases}$$

Then $g^{(n)}(0) = 0$ for all $n \in \mathbb{N}$. Then, the Taylor series for $g(x)$ converges but not to $g(x)$ except at $x = 0$.

Chapter 7

The Riemann Integral

7.1 Discussion: How Should Integration be Defined?

7.2 The Definition of the Riemann Integral

Definition 7.2.1: Definition 7.2.1.

A **partition** P of $[a,b]$ is a finite set of points from $[a,b]$ that includes both a and b. The notational convention is to always list the points of a partition $P = \{x_0, x_1, x_2, \ldots, x_n\}$ in increasing order; thus,

$$a = x_0 < x_1 < x_2 < \cdots < x_n = b.$$

For each subinterval $[x_{k-1}, x_k]$ of P, let

$$m_k = \inf\{f(x) : x \in [x_{k-1}, x_k]\} \quad \text{and} \quad M_k = \sup\{f(x) : x \in [x_{k-1}, x_k]\}.$$

The **lower sum** of f with respect to P is given by

$$L(f,P) = \sum_{k=1}^{n} m_k (x_k - x_{k-1}).$$

Likewise, we define the **upper sum** of f with respect to P by

$$U(f,P) = \sum_{k=1}^{n} M_k (x_k - x_{k-1}).$$

Definition 7.2.2: Definition 7.2.2.

A partition Q is a **refinement** of a partition P if Q contains all of the points of P; that is, if $P \subseteq Q$.

Lemma 7.2.1 Lemma 7.2.3.

If $P \subseteq Q$, then $L(f,P) \leqslant L(f,Q)$, and $U(f,P) \geqslant U(f,Q)$.

Lemma 7.2.2 Lemma 7.2.4.

If P_1 and P_2 are any two partitions of $[a,b]$, then $L(f,P_1) \leqslant U(f,P_2)$.

7.2.1 Integrability

Definition 7.2.3: Definition 7.2.5.

Let $\mathcal{P}$ be the collection of all possible partitions of the interval $[a, b]$. The **upper integral** of f is defined to be

$$U(f) = \inf\{U(f, P) : P \in \mathcal{P}\}$$

In a similar way, define the **lower integral** of f by

$$L(f) = \sup\{L(f, P) : P \in \mathcal{P}\}$$

Lemma 7.2.3 Lemma 7.2.6.

For any bounded function f on $[a, b]$, it is always the case that $U(f) \geqslant L(f)$.

Lemma 7.2.4 Definition 7.2.7 (Riemann Integrability)

A bounded function f defined on the interval $[a, b]$ is **Riemann-integrable** if $U(f) = L(f)$. In this case, we define $\int_a^b f$ or $\int_a^b f(x)dx$ to be this common value; namely,

$$\int_a^b f = U(f) = L(f).$$

7.2.2 Criteria for Integrability

Theorem 7.2.1 Theorem 7.2.8 (Integrability Criterion)

A bounded function f is integrable on $[a, b]$ if and only if, for every $\epsilon > 0$, there exists a partition P_ϵ of $[a, b]$ such that

$$U(f, P_\epsilon) - L(f, P_\epsilon) < \epsilon.$$

Theorem 7.2.2 Theorem 7.2.9.

If f is continuous on $[a, b]$, then it is integrable.

7.3 Integrating Functions with Discontinuities

Theorem 7.3.1 Theorem 7.3.2.

If $f : [a, b] \to \mathbf{R}$ is bounded, and f is integrable on $[c, b]$ for all $c \in (a, b)$, then f is integrable on $[a, b]$. An analogous result holds at the other endpoint.

7.4 Properties of the Integral

Theorem 7.4.1 Theorem 7.4.1.

Assume $f : [a, b] \to \mathbf{R}$ is bounded, and let $c \in (a, b)$. Then, f is integrable on $[a, b]$ if and only if f is integrable on $[a, c]$ and $[c, b]$. In this case, we have

$$\int_a^b f = \int_a^c f + \int_c^b f$$

Theorem 7.4.2 Theorem 7.4.2.
Assume f and g are integrable functions on the interval $[a, b]$.

(a) The function $f + g$ is integrable on $[a, b]$ with $\int_a^b (f + g) = \int_a^b f + \int_a^b g$.

(b) For $k \in \mathbf{R}$, the function kf is integrable with $\int_a^b kf = k \int_a^b f$.

(c) If $m \leqslant f(x) \leqslant M$ on $[a, b]$, then $m(b - a) \leqslant \int_a^b f \leqslant M(b - a)$.

(d) If $f(x) \leqslant g(x)$ on $[a, b]$, then $\int_a^b f \leqslant \int_a^b g$.

(e) The function $|f|$ is integrable and $\left|\int_a^b f\right| \leqslant \int_a^b |f|$.

Theorem 7.4.3 Definition 7.4.3.
If f is integrable on the interval $[a, b]$, define

$$\int_b^a f = -\int_a^b f.$$

Also, for $c \in [a, b]$ define

$$\int_c^c f = 0.$$

7.4.1 Uniform Convergence and Integration

Theorem 7.4.4 Theorem 7.4.4 (Integrable Limit Theorem)
Assume that $f_n \to f$ uniformly on $[a, b]$ and that each f_n is integrable. Then, f is integrable and

$$\lim_{n\to\infty} \int_a^b f_n = \int_a^b f$$

7.5 The Fundamental Theorem of Calculus

Theorem 7.5.1 Theorem 7.5.1 (Fundamental Theorem of Calculus)

(a) If $f : [a, b] \to \mathbf{R}$ is integrable, and $F : [a, b] \to \mathbf{R}$ satisfies $F'(x) = f(x)$ for all $x \in [a, b]$, then

$$\int_a^b f = F(b) - F(a).$$

(b) Let $g : [a, b] \to \mathbf{R}$ be integrable, and for $x \in [a, b]$, define

$$G(x) = \int_a^x g.$$

Then G is continuous on $[a, b]$. If g is continuous at some point $c \in [a, b]$, then G is differentiable at c and $G'(c) = g(c)$.

7.6 Lebesgue's Criterion for Riemann Integrability

7.6.1 Riemann-integrable Functions with Infinite Discontinuities

Recall from Section 4.1 that Thomae's function

$$t(x) = \begin{cases} 1 & \text{if } x = 0 \\ 1/n & \text{if } x = m/n \in \mathbf{Q}\backslash\{0\} \text{ is in lowest terms with } n > 0 \\ 0 & \text{if } x \notin \mathbf{Q} \end{cases}$$

is continuous on the set of irrationals and has discontinuities at every rational point. Thomae's function is integrable on $[0,1]$ with $\int_0^1 t = 0$.

7.6.2 Sets of Measure Zero

Definition 7.6.1: Definition 7.6.1.

A set $A \subseteq \mathbf{R}$ has measure zero if, for all $\epsilon > 0$, there exists a countable collection of open intervals O_n with the property that A is contained in the union of all of the intervals O_n and the sum of the lengths of all of the intervals is less than or equal to ϵ. More precisely, if $|O_n|$ refers to the length of the interval O_n, then we have

$$A \subseteq \bigcup_{n=1}^{\infty} O_n \quad \text{and} \quad \sum_{n=1}^{\infty} |O_n| \leqslant \epsilon$$

7.6.3 α-Continuity

Definition 7.6.2: Definition 7.6.3.

Let f be defined on $[a,b]$, and let $\alpha > 0$. The function f is **α-continuous** at $x \in [a,b]$ if there exists $\delta > 0$ such that for all $y, z \in (x-\delta, x+\delta)$ it follows that $|f(y) - f(z)| < \alpha$.
Let f be a bounded function on $[a,b]$. For each $\alpha > 0$, define D^α to be the set of points in $[a,b]$ where the function f fails to be α-continuous; that is,

$$D^\alpha = \{x \in [a,b] : f \text{ is not } \alpha\text{-continuous at } x.\}$$

Definition 7.6.3

For a fixed $\alpha > 0$, a function $f : A \to \mathbf{R}$ is **uniformly α-continuous** on A if there exists a $\delta > 0$ such that whenever x and y are points in A satisfying $|x - y| < \delta$, it follows that $|f(x) - f(y)| < \alpha$. By imitating the proof of

7.6.4 Compactness Revisited

Theorem 7.6.1 Theorem 7.6.4.

Let $K \subseteq \mathbf{R}$. The following three statements are all equivalent, in the sense that if any one is true, then so are the two others.

(a) Every sequence contained in K has a convergent subsequence that converges to a limit in K.

(b) K is closed and bounded.

(c) Given a collection of open intervals $\{G_\lambda : \lambda \in \Lambda\}$ that covers K (that is, $K \subseteq \bigcup_{\lambda \in \Lambda} G_\lambda$) there exists a finite subcollection $\{G_{\lambda_1}, G_{\lambda_2}, G_{\lambda_3}, \ldots, G_{\lambda_N}\}$ of the original set that also covers K.

7.6.5 Lebesgue's Theorem

Theorem 7.6.2 Theorem 7.6.5 (Lebesgue's Theorem)

Let f be a bounded function defined on the interval $[a,b]$. Then, f is Riemann-integrable if and only if the set of points where f is not continuous has measure zero.